PHYSICAL REVIEW E **67**, 025601(R) (2003)

Dark and gray strong dispersion-managed solitons

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(Received 18 July 2002; published 12 February 2003)

Dark and gray solitons in communication systems with strong dispersion management (DM) are obtained. These new modes are characterized by a decaying oscillatory background. Unlike the bright DM solitons in which the oscillations are observed only on a logarithmic scale, here the oscillations are dominant on a linear scale and become very strong for moderate map strength.

DOI: 10.1103/PhysRevE.67.025601 PACS number(s): 42.65.Tg

In recent years researchers have developed an important technology referred to as dispersion management (DM). DM means that optical fibers with sharply different dispersion characteristics, anomalous and normal, are combined together in subsections of the fiber and then this substructure is repeated periodically to make up the entire fiber length. The relevant fundamental equation governing the dynamics is the nonlinear Schrödinger equation (NLSE) [1,2]. But now the dispersion is a function of distance along the fiber, $D=D(z)$, which is composed of an average term plus a large rapidly varying (periodic) function. This equation admits soliton type solutions, called DM solitons [3,4]. Such modes are less susceptible to Gordon-Haus jitter [5,6] and in the WDM context have substantially reduced FWM components than their classical counterparts. As demonstrated in recent transmission experiments [7,8], DM solitons are amongst the transmission formats being actively considered for the next generation optical communications systems.

To date, all strong DM soliton research has focused on "bright" solitons; a bright soliton being one that vanishes well away from its peak (center) point, e.g., a sech profile. However in the classical case, it is also well known that in the normal regime, dark and gray solitons exist [9,10]. The intensity of these dark and gray solitons tends to a nontrivial background state away from its center point. Indeed for the dark and gray classical solitons, the "center" point is located at the minimum in amplitude, unlike the bright case, where the center point is a maximum. There has been substantial research, both analytical and experimental, investigating such dark and gray "classical" solitons [11]. It is natural, therefore, to consider dark and gray strong DM solitons. We also note that in the *weak* DM case, it is straightforward to show that the dynamics of dark solitons reduces to the classical case [12].

In this paper, we show how to obtain dark and gray strong DM solitons and investigate their properties. By considering the perturbed NLSE with loss and lumped amplification, we derive in the limit of strong dispersion management, an averaged equation governing the slow dynamics of the optical field's amplitude. Stationary dark and traveling gray solitons are obtained by using asymptotic analysis and direct averaging procedures. The DM dark and gray modes are found to be very different from the classical ones. The most important distinction is that they are characterized by a decaying oscillatory background. Unlike the corresponding bright DM solitons in which the oscillations are observed only on a logarithmic scale, here the oscillations are dominant on a linear scale and become very strong even for moderate map strength. The propagation of an optical pulses in dispersion managed fibers is described by the NLSE:

$$i\frac{\partial u}{\partial z}-\frac{D(z)}{2}\frac{\partial^2 u}{\partial t^2}+g(z)|u|^2u=0, \tag{1}$$

where u is related to the slowly varying envelope of the electric field, z to the propagation distance, and t to the retarded time. All dimensionless quantities z, t, and u are related to the actual physical variables [1,2]. The functions $D(z)$ and $g(z)$ describe the local group velocity dispersion of the fiber and the variation of power due to loss and lumped amplifications, respectively. Both functions are taken to be periodic with period z_a, which measures the dimensionless distance between amplifiers. In this paper we will consider the lossless: $g=1$, and lossy case for which $g(z)=g_0\exp[-2\Gamma(z-nz_a)]$ for $nz_a\leqslant z\leqslant(n+1)z_a$, where $g_0=2\Gamma z_a/[1-\exp(-2\Gamma z_a)]$ and Γ is the dimensionless loss coefficient. When the dispersion coefficient is large and changes rapidly with z_a, Eq. (1) is reduced to a nonlocal integral equation which admits a bright soliton solution [4]. However, so far the question, how to obtain strong DM dark and gray solitons, is still open. Here, we present such solutions and the essential analysis. We consider the case in which the dispersion is a large periodic function with period z_a and varies rapidly, i.e., $D(z)=\delta_a+(1/z_a)\Delta(z/z_a)$, where δ_a is the average dispersion and Δ is periodic in z_a with average zero. We look for a solution of the form

$$u=[u_{-\infty}+U(z,t)]\exp\left(i\lambda^2\int_0^z g(z')dz'\right) \tag{2}$$

with $u\to u_{\pm\infty}$ (real) as $t\to\pm\infty$ and $U(z,t)$ is a complex amplitude. Here, $\lambda^2=u_{-\infty}^2=u_{+\infty}^2$ is the propagation constant. Substituting Eq. (2) into Eq. (1) we find

$$i\frac{\partial U}{\partial z}-\frac{D(z)}{2}\frac{\partial^2 U}{\partial t^2}+\lambda^2 g(U+U^*)+gu_{-\infty}(2|U|^2+U^2)+g|U|^2U=0. \tag{3}$$

Our approach to solve Eq. (3) is based on Fourier transform methods [4,13,14]. However, since U does not vanish at infinity, we cannot apply the Fourier transform directly on Eq.

(3). To overcome this difficulty, we take the time derivative of Eq. (3) and get the following:

$$i\frac{\partial V}{\partial z}-\frac{D(z)}{2}\frac{\partial^2 V}{\partial t^2}+2gu_{-\infty}(UV+VU^*+UV^*)$$
$$+g(2|U|^2V+U^2V^*)+\lambda^2 g(V+V^*)=0, \qquad (4)$$

with $V\equiv\partial U/\partial t$. It is evident that for a dark soliton, V vanishes at the boundaries, i.e., it forms a localized function. Since in this case, Eq. (4) contains both slowly and rapidly varying terms, we introduce new fast and slow scales as $\zeta = z/z_a$ and $Z=z$, respectively and expand U and V in powers of z_a, i.e., $U=U^{(0)}+z_aU^{(1)}+\cdots$, $V=V^{(0)}+z_aV^{(1)}+\cdots$. Substituting the above expansion together with the dispersion map into Eq. (4) we find that the leading order equation in $1/z_a$ is $\mathcal{J}(V^{(0)})=0$ and the order 1 equation is $\mathcal{J}(V^{(1)})= -\mathcal{R}$, where $\mathcal{J}(A)\equiv i\partial A/\partial\zeta-(\Delta(\zeta)/2)\partial^2A/\partial t^2$ and $\mathcal{R}\equiv\mathcal{R}_{\rm L}+\mathcal{R}_{\rm NL}$ with $\mathcal{R}_{\rm L}$ and $\mathcal{R}_{\rm NL}$ being the linear and nonlinear inhomogeneous parts, respectively given by

$$\mathcal{R}_{\rm L}=i\frac{\partial V^{(0)}}{\partial Z}+\lambda^2 g(\zeta)(V^{(0)}+V^{(0)*})-\frac{\delta_a}{2}\frac{\partial^2 V^{(0)}}{\partial t^2},$$

$$\mathcal{R}_{\rm NL}=2g(\zeta)u_{-\infty}(V^{(0)}U^{(0)*}+U^{(0)}V^{(0)*}+U^{(0)}V^{(0)})$$
$$+g(\zeta)(2|U^{(0)}|^2V^{(0)}+U^{(0)2}V^{(0)*}). \qquad (5)$$

To solve at order $1/z_a$, we use the Fourier transform

$$\hat f(\omega)\equiv\mathcal{F}(f)=\int_{-\infty}^{+\infty}dt\ e^{-i\omega t}f(t),$$

$$f(t)\equiv\mathcal{F}^{-1}(\hat f)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}d\omega\ e^{i\omega t}\hat f(\omega). \qquad (6)$$

The solution is thus given in the Fourier representation by $\hat V^{(0)}=\hat\phi(\omega,Z)\exp[i\omega^2C(\zeta)/2]$, where $C(\zeta)=\int_0^\zeta\Delta(\zeta')d\zeta'$. The amplitude $\hat\phi(\omega,Z)$ is an arbitrary function whose dynamical evolution is determined by a secularity condition associated with the order 1 equation. In other words, the condition of the orthogonality of $\mathcal{R}$ to all eigenfunctions of the adjoint linear problem which, when written in the Fourier domain, takes the form

$$\int_0^1 d\zeta\mathcal{F}(\mathcal{R})\exp[-i\omega^2C(\zeta)/2]=0. \qquad (7)$$

Substituting the expression for $V^{(0)}=\mathcal{F}^{-1}(\hat V^{(0)})$ into $\mathcal{R}$ and performing the integration in condition (7) yields the following nonlinear evolution equation:

$$i\frac{\partial\hat\phi(\omega)}{\partial Z}+\left(\lambda^2+\frac{\delta_a\omega^2}{2}\right)\hat\phi(\omega)+\lambda^2\mathcal{K}(\omega)[\hat\phi(-\omega)]^*$$
$$+\langle\mathcal{R}_{\rm NL}\rangle(\omega)=0, \qquad (8)$$

where $\mathcal{K}(\omega)$ and $\langle\mathcal{R}_{\rm NL}\rangle$ depend on the shape of the dispersion map under consideration and are given by

$$\mathcal{K}(\omega)\equiv\int_0^1 d\zeta\ g(\zeta)\exp[-i\omega^2C(\zeta)],$$

$$\langle\mathcal{R}_{\rm NL}\rangle(\omega)\equiv\int_0^1 d\zeta\ \mathcal{F}(\mathcal{R}_{\rm NL})\exp[-i\omega^2C(\zeta)/2]. \qquad (9)$$

We call Eq. (8) the dispersion managed nonlinear Schrödinger equation for a nonzero background which governs the averaged evolution (in Fourier space) of an optical beam in the regime of strong dispersion. The above results hold for any periodic dispersion map. However, the analysis simplifies significantly in the special case of a two-step dispersion map for which two fiber segments with different dispersion coefficients are fused in every period. In this case we have $\Delta(\zeta)=\Delta_1$ for $0\leqslant|\zeta|<\theta/2$ and Δ_2 in the region $\theta/2<|\zeta|<1/2$, where θ is the fraction of the map with dispersion Δ_1. For the lossless case $[g(\zeta)=1]$ the kernel $\mathcal{K}(\omega)$ takes the simple form of $\mathcal{K}_{\rm lossless}=\sin(s\omega^2)/(s\omega^2)$, with $s=[\theta\Delta_1-(1-\theta)\Delta_2]/4$ which provides a measure of the normalized map strength. Next, we look for a Z independent solution to Eq. (8) in the form $\hat\phi(\omega,Z)=\hat\phi_s(\omega)$ (real and even), which when inserted into Eq. (8) leads to

$$\hat\phi_s(\omega)=-\frac{\langle\mathcal{R}_{\rm NL}\rangle(\omega)}{\lambda^2+\lambda^2\mathcal{K}(\omega)+\delta_a\omega^2/2}\equiv\mathcal{M}[\hat\phi_s(\omega)]. \qquad (10)$$

Note that when $s\to0$ $(\mathcal{K}\to1)$, the problem reduces to finding "classical" dark solitons. More general solutions that depend on Z are also possible and describe "breathing" modes. To find the mode shape, we employ a modified Neumann iteration scheme and write Eq. (10) in the form

$$\hat\phi_s^{(m+1)}(\omega)=(s_L/s_R)^2\mathcal{M}[\hat\phi_s^{(m)}(\omega)],\quad m\geqslant0, \qquad (11)$$

with the convergence factors $s_L=\int|\hat\phi_s^{(m)}(\omega)|^2\,d\omega$ and $s_R=\int\hat\phi_s^{(m)}(\omega)\mathcal{M}\,d\omega$. To implement the above algorithm, we start with an initial guess for $\hat\phi_s(\omega)$ which is even e.g., $\hat\phi_s(\omega)^{\rm initial}=a/\cosh(b\omega)$ and directly obtain the solution $\hat V^{(0)}$. By applying the inverse Fourier transform on $\hat V^{(0)}_{\rm initial}(\omega)$, we obtain the mode $V^{(0)}_{\rm initial}(t)$ in physical space. Then the initial guess for the solution $U^{(0)}(t)$, taking $U^{(0)}(-\infty)=0$, follows by integration over $V^{(0)}_{\rm initial}(t)$. Once the values of both initial guesses are obtained, the evaluation of the nonlinear term $\mathcal{R}_{\rm NL}$ follows from Eq. (5). The last step is to take the Fourier transform of $\mathcal{R}_{\rm NL}$ and perform the integral in Eq. (9). In Figs. 1 and 2 we show typical examples of dark soliton solutions obtained for the lossless case with $u_{-\infty}=-u_{+\infty}=-1$, $\lambda^2=1$ and for different values of map strength. Importantly, unlike the bright DM case in which the beam's core profile is close to a Gaussian shape, here, the center of the dark DM mode is close to the classical case. Remarkably, we find that the dark DM mode exhibits strong oscillations on a linear scale even for map strength $s\approx1$. This is in distinction with the bright DM case in which the oscillations

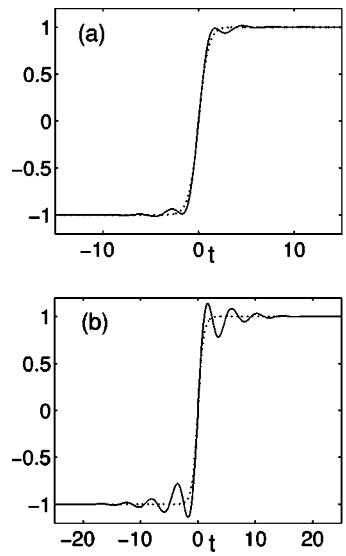

FIG. 1. Amplitude of the DM dark soliton (solid line) at zero chirp point obtained for the lossless case with $u_{-\infty}=-u_{+\infty}=-1$, $\lambda^2=1$, $\delta_a=1$, and $z_a=0.1$. (a) $\Delta_1=-\Delta_2=4$, $s=1$; (b) $\Delta_1 bf=-\Delta_2=6$, $s=1.5$. The dotted line shows the classical dark soliton.

are noticeable only on a logarithmic scale (for the same value of s). Moreover, we find that for moderate values of map strengths ($s\gtrsim 2$) the oscillations become very large. To understand the origin of these oscillations, we note that $|\mathcal{K}(\omega)|<1$ in which case Eq. (10) takes the form

$$\hat{\phi}_s(\omega)=-\frac{\langle\mathcal{R}_{\mathrm{NL}}\rangle(\omega)}{\lambda^2+\delta_a\omega^2/2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\lambda^2\mathcal{K}(\omega)}{\lambda^2+\delta_a\omega^2/2}\right)^n. \tag{12}$$

To leading order in $\mathcal{K}(\omega)$, the above equation reads

$$\hat{\phi}_s(\omega)\approx-\frac{\langle\mathcal{R}_{\mathrm{NL}}\rangle(\omega)}{\lambda^2+\delta_a\omega^2/2}\left[1-\frac{\lambda^2\mathcal{K}(\omega)}{\lambda^2+\delta_a\omega^2/2}\right]. \tag{13}$$

Scrutinizing Eq. (13) we conclude that the origin of the oscillatory behavior of the dark soliton is the presence of the term $\mathcal{K}(\omega)$. For the classical case, $\mathcal{K}(\omega)\rightarrow 1$ and therefore, the oscillations disappear. However for moderate map strength, the oscillations become significant due to the behavior of $\mathcal{K}(\omega)$. Next we compare our results with Ref. [15]. In Ref. [15], an approximate form for a DM dark soliton was obtained as $|u_{Dark,DM}|\approx|1-\exp(-2t^2)|$. One way to understand such an approximation is as follows. For the classical dark soliton we know that $|u_{Dark,C}|^2=1-\mathrm{sech}^2(t)=1-|u_{Bright,C}|^2$. If one extends this idea to the DM case then $u_{Bright,C}\rightarrow u_{Bright,DM}$. We know, however, that for a moderate value of map strength s a good approximation for $u_{Bright,DM}$ is $\exp(-t^2)$ and hence the approximation above for dark DM follows. However, this approximation fails for large map strength, where oscillations on the tails begin to grow. In fact, as indicated in our analysis for the dark DM

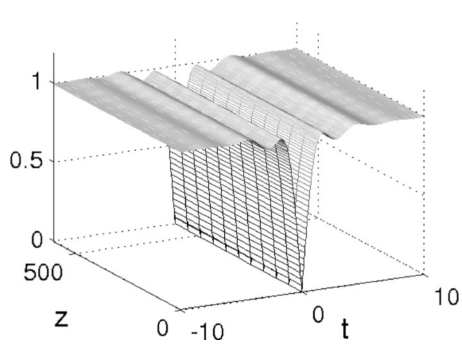

FIG. 2. Stationary evolution of the DM dark soliton computed at the end of each dispersion map for the parameters depicted in Fig. 1(a).

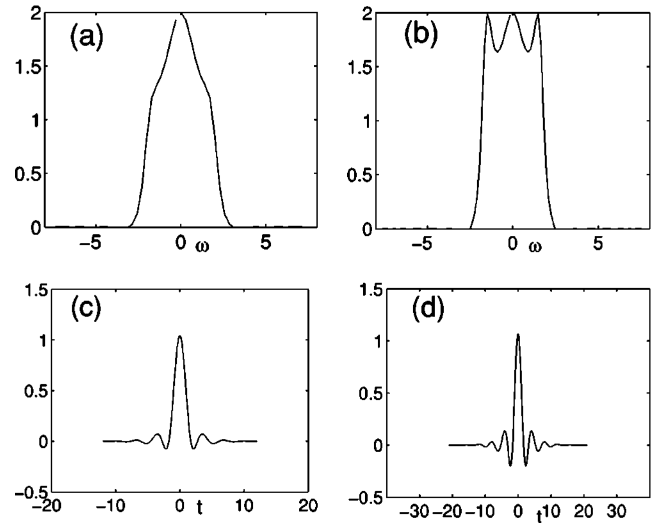

FIG. 3. $\hat{\phi}_s(\omega)$ for $s=1$ (a) and $s=1.5$ (b) as well as the leading order solution, $V^{(0)}(t)=\mathcal{F}^{-1}(\hat{V}^{(0)})$, in the time domain computed at $z=0$ for $s=1$ (c) and $s=1.5$ (d). Parameters are the same as in Fig. 1.

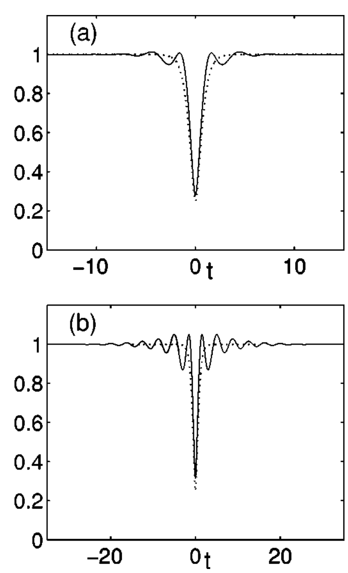

FIG. 4. Intensity of a DM gray soliton mode (solid line) at zero chirp point obtained for the lossless case with $u_{-\infty}=-u_{+\infty}=-1$, $\lambda^2=1$, $\delta_a=1$, $z_a=0.1$, $\alpha=0.25$. (a) $\Delta_1=-\Delta_2=4$, $s=1$; (b) $\Delta_1=-\Delta_2=6$, $s=1.5$. The dotted line depicts the classical gray soliton solution.

case, the oscillations grow more rapidly than the bright case as a function of s (see Fig. 3).

We next compare our results with direct averaging methods cf. [16] applied on Eq. (4) for the lossless case. Initially, we start from a guess, say, $V_0 = \mathrm{sech}(t)$ and $U_0 = 1 + \tanh(t)$ with fixed energy $\mathcal{E}_0 = \int_{-\infty}^{+\infty} |V_0|^2 \, dt$. Over one period, this initial ansatz will evolve into V_0' which in general will have a chirp. Next, we define the average beam $V_0'' = (V_0 + |V_0'|)/2$ with energy $\mathcal{E}_0''$. Then $V_1 = V_0''(\mathcal{E}_0/\mathcal{E}_0'')^{1/2}$ will be the new initial guess. At each step, the function U is obtained by $U = \int_{-\infty}^{t} V(\tau) d\tau$. The above procedure is repeated until we converge to the exact solution. Note that for small value of z_a, the asymptotic analysis is in good agreement with the direct averaging method (the difference is of order 10^{-2}, not noticeable in Fig. 1). We also obtained modes for the lossy case ($g \neq 1$); it will be reported elsewhere.

Next, we mention briefly how to obtain gray DM solitons in which the minimum intensity does not vanish. The main idea is to look for solutions of the form

$$u(z,t) = u_0[U(\xi,z) + \mu]\exp\left[i\lambda^2 \int_0^z g(z')dz' \right], \quad (14)$$

where $\xi = u_0\beta[t + u_0\alpha\int_0^z D(z')dz']$ with u_0^2 being the intensity far from the beam center. Here, $\mu = i\alpha - \beta$ and U is a complex valued amplitude. The solution is supplemented with the following boundary conditions: $\mathrm{Re}(U) \to 2\beta$ as $\xi \to +\infty$, $\mathrm{Re}(U) \to 0$ as $\xi \to -\infty$, $\mathrm{Im}(U) \to 0$ as $\xi \to \pm\infty$ with $|\mu|^2 = 1$. The intensity of the beam far away from the center is u_0^2 and as a result the propagation constant is given by $\lambda^2 = u_0^2$. Substituting Eq. (14) into Eq. (1), and taking the derivative of the resulting equation with respect to ξ we find

$$i\frac{\partial V}{\partial z} + u_0^2 D(z)\left(i\alpha\beta\frac{\partial V}{\partial \xi} - \frac{\beta^2}{2}\frac{\partial^2 V}{\partial \xi^2} \right) + u_0^2 g(V + \mu^2 V^*) + N(U,V) = 0, \quad (15)$$

where $V = \partial U/\partial \xi$ and $N(U,V) = u_0^2 g[2\mu^* UV + 2\mu(U^*V + UV^*) + 2|U|^2 V + U^2 V^*]$. Following the same procedure described before, we obtain averaged equations governing the evolution of the beam amplitude in Fourier space. Figure 4 shows typical gray solitons obtained by this procedure for the lossless case. Notice that apart from the oscillatory characteristic, gray DM modes have higher minimum intensity than their classical counterpart.

M.J.A. is partially supported by the NSF under Grant Nos. ECS-9800152 and DMS-0070792.

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